CONCEPTUAL CHANGE IN ADVANCED MATHEMATICAL THINKING

Irene Biza, **Alkeos Souyoul** and **Theodossios Zachariades**

Department of Mathematics, University of Athens

ABSTRACT

In this paper, we argue that the theoretical framework of conceptual change could help us to interpret some of the misconceptions dealing with concepts of advanced mathematical thinking, as the concept of curves´ tangent, which the students have studied in specific cases in the middle high school and they deal with the general case of them in the upper secondary and tertiary level. In this study, we trace the beliefs of the students and the synthetic models, which they create in their effort to assimilate the general concept of curves´ tangent in their existing knowledge of the tangent of circle and conic sections. We make a case that students take for granted properties of circle's tangent in curves, which do not apply in general and they cause the synthetic model that students create to deal with tangent's problems.

INTRODUCTION

This study is based on the theory of the *Conceptual Change*. This theory examines the learning process, especially in cases where the new knowledge is incompatible with the prior one (Vosniadou & Brewer, 1992; Vosniadou, 1994). According to this theory, the students very early create initial explanatory frameworks that consist of certain coherent core of *presuppositions*. These *presuppositions* influence *beliefs* that are created through every day and cultural experience. When students face a new knowledge, which is incompatible with the prior one, in their effort to assimilate the new information in their existing cognitive base, they create *synthetic models*, which are a mixture of their existing *beliefs* and the scientific theory.

In this study, we examine the learning development of the notion of curves´ tangent from this theory point of view. We investigate the *beliefs* of the students and the *synthetic models*, which they create in their effort to assimilate the generalized concept of curves´ tangent in their existing knowledge concerning tangent of circle and conic sections.

We make a case that students take for granted properties of circle's tangent in curves, which do not apply in general. These properties are the *generic properties* of the corresponding *concept image* (Tall & Vinner, 1981; Tall, 1986; Vinner, 1991). It seems that students generate a *paradigmatic intuitive model* of circle (Fischbein, 1987). This model remains active after the traditional instruction of the general case of this concept in the upper high school and causes the *synthetic models* that students create to deal with tangent's problems.

THEORETICAL FRAMEWORK

The theory of *conceptual change* examines the process of knowledge acquisition and especially in situations where the prior knowledge is incompatible with the new. According to this theory, the children, in their effort to understand the world around them, create a *framework theory*. This is not a formal theory but something like a *naive* theory, that is an explanatory framework created from first ages and it is consisted of ontological and epistemological *presuppositions* structured in a coherent core. These *presuppositions* are influenced by everyday experience. In most of the cases, students are not aware of the control of the constraints of these *presuppositions* in their interpretation of receiving information and their conceptualization. This *framework theory*, through everyday and cultural experience, causes some *specific theories* (Vosniadou & Brewer, 1992; Vosniadou, 1994). The *beliefs* that constitute a *specific theory*, act as a secondary level of constraints in the process of knowledge acquisition. These *beliefs* and the *presuppositions* that cause them are *intuitive knowledge* with the meaning that Fischbein (1987) gave to the notion.

Many times, these existing *presuppositions* and *beliefs* influence the acquisition of new knowledge and cause cognitive problems. *Conceptual change* theory tries to interpret exactly these problems. In many cases, the new information is incompatible with the existing *presuppositions* and *beliefs* of the student. In these cases, the acquisition of new information needs a radical revision of prior knowledge. In fact, it needs a radical *conceptual change* that is a difficult and time-consuming process of learning. Usually, the students' *beliefs* according to their intuitive nature are too strong and consistent. Consequently, various failures occur in the learning process and some of these create misconceptions that take place in a not arbitrary way. The *synthetic model* is of this kind of misconceptions. The term of model is used as the *mental models*, which is a mental representation generated by a person during his/her cognitive operations when he/she confronts a problematic situation. Especially, the *synthetic model* is a model that reveals students' misconceptions when they try to reconcile new information with their initial *explanatory theory*. These models are a mixture of existing beliefs of individuals and the scientific knowledge concerning the same notion. Actually, the students create *synthetic models* in their effort to assimilate the new information in their existing cognitive base although they are incompatible (Vosniadou & Brewer, 1992; Vosniadou, 1994). Examples of such synthetic models, in the case of science, is the model of the Earth as "a hollow sphere with people living inside it on flat ground" (Vosniadou & Brewer, 1992; Vosniadou, 1994) or, in the case of mathematics, is the model of a fraction as a part of the unit where "the more parts means the less value" (Stafilidou & Vosniadou, 2004).

The theory of *conceptual change* has already applied to a considerable number of cases of science learning. In addition, some recent studies investigate *conceptual change* in the learning process of mathematical concepts. These are referred to the concept of number (Merenluoto & Lehtinen, 2002); to the transition from one set of numbers to a more extensive one (eg. from natural numbers to fractions or rational numbers) (Stafilidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2002, 2004a, 2004b); to proportion (Van Dooren, De Bock, Hessels, Janssens, Verschaffel, 2004) and to infinity (Hannula, Markku, Maijala, Pehkonen, & Soro, 2002; Tirosh & Tsamir, 2004). Many other recherchers have investigated students' previous conceptions concerning mathematical notions and their incompatibility with the corresponding formal knowledge. Fischbein (1987) talked about *intuitions* and their effects in mathematical reasoning, Vergnaud (1988, 1990) mentioned the existence of implicit mathematical concepts and theorems which act as invariants and called them *concepts-in-action* and *theorems-in-action*, Cornu (1991) described *spontaneous conceptions* before formal thinking, Stavy and Tirosh (2000) expounded their theory of *intuitive rules*. Conceptual change approach does not contravene the above theories but it offers a social constructivism perspective and tries to provide, among others, student-centered explanations about knowledge acquisition concerning counter intuitive math concepts and to alert students against the use of additive mechanisms in these cases (Vosniadou, 2004).

In this study, we examine the learning development of the notion of curves΄ tangent. The students have studied the concept of the tangent of circle in middle high school. In upper high school, they deal with the tangent of conic sections and later on with the tangent of a curve. The historical analysis of this notion reveals its different aspects as they appeared through the evolution of mathematics science. This historical path could give us a support in our effort to interpret certain answers of students, which could make known their conceptions about tangent line (Artigue, 1990).

The aim of this paper is to interpret the students' misconceptions concerning tangent line from the *conceptual change* point of view. We trace the *beliefs* of the students and the *synthetic models*, which they create in their effort to assimilate the general concept of curves΄ tangent in their existing knowledge of the tangent of circle and conic sections.

METHODOLOGY

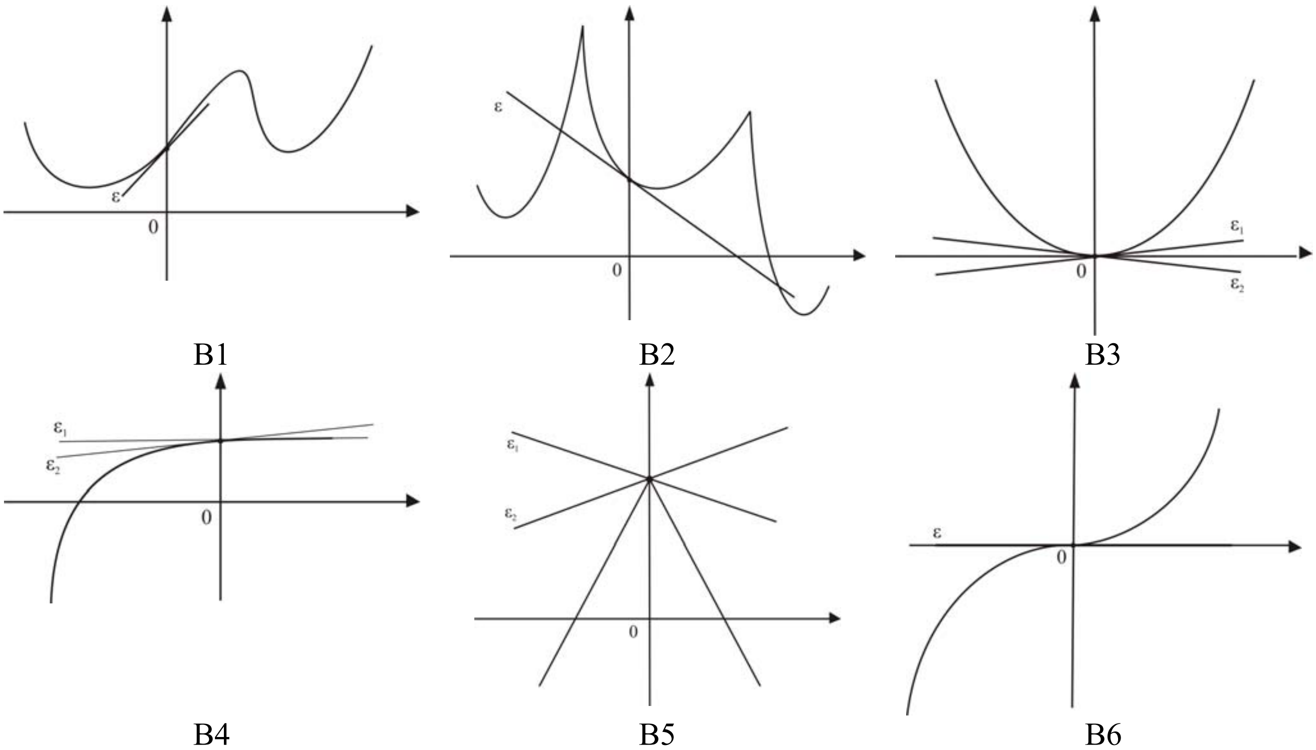
The participants of this study were 19 first year university students of mathematics, of various levels of performance. They answered a questionnaire printed on paper and afterwards we had an interview with each one of them and discussed his/her answers. All the conversations were recorded during the interviews. Through the previous year, all of these students had a traditional calculus course at high school, which included the concepts of limit, continuity, derivative, tangent line and integral. By the time we interviewed them, they had not been taught these concepts at a university level.

The questionnaire included three parts. The tasks of the first part aimed to investigate student's beliefs about the properties of a tangent. The tasks of the second part aimed to investigate the student's ability to recognize a tangent. The tasks of the third part aimed to test the validity of students' answers in the second part and the persistence of their mistakes.

In the first part of the questionnaire, the students were asked to determine whether the following statements are True of False:

> **A1**: *The tangent of a curve at a point x_0, is a line having exactly one point common with the curve and it does not split it.*
>
> **A2**: *The tangent of a curve at a point x_0, divides the plane into two semi-planes one of which contains the whole curve.*
>
> **A3**: *The tangent of a curve at a point x_0, may have more than one point common with the curve.*

In the second part of the questionnaire, the students were asked to determine which of the drawn lines in the following figure are tangent at point A(0,f(0)).



Figure

In the third part of the questionnaire the students were asked to draw the tangent at a point, if there is one, of some curves, similar but not the same as those in the second part. The students answered in this part in a similar way to the second part. They did not draw any unpredictable tangent. So, we do not present here the third part of the questionnaire as it does not provide anything more except of the verification of the results of the second part.

FINDINGS

According to the students' answers to the questionnaire and the interviews, three classes were defined, depending on the extent that the elementary definition of circle's tangent dominates their *concept images* about tangent (table).

First class	Second class	Third class
"circle concept image"	*"circle-like concept image"*	*"curve concept image"*
5	7	7
26%	37%	37%

Table: Findings

The first class comprised five (5/19) students. They had a *"circle concept image"* of tangent. These students, generally, gave wrong answers to the tasks of the first part and they used the properties of circle's tangent to identify the tangent in the tasks of the second part of the questionnaire. Some of them accepted the line ε in B1 graph of the figure as a tangent, but they rejected the line ε in the second graph. In the interviews, they explained their choice by mentioning that in the first graph the other common points are not on view in the figure while in the second they are. The following dialogue indicates an explanation that was given in the interviews:

S: In B2 the line crosses the curve and it intersects the curve more than once… but in B1 it does not.

I: Then what will happen if we extend the line in B1. Won't it look like the one in B2?

S: Can we do that? If this is happened it will no longer be a tangent. But this is not the case. No one did extend it.

Many of these students, also, rejected other correct tangents, as x-axis in B6 graph, which splits the curve while they accepted, as tangents, lines that are not, as ε_1 or ε_2 in B3 graph. For some of them, B5 graph makes an exception to this. The point A is a "corner point" and two of these students remembered that "if the common point is a corner one there is no tangent", so they rejected it without being able to say why (this was the beginning of a fruitful conflict).

The second class comprised seven (7/19) students. They had created a more sophisticated *concept image* of tangent. We will call this *"circle-like concept image"*. They checked the validity of circle's properties locally. For them: "a curve has a tangent at a point, if there is a neighbourhood around this point, where the curve seems like a circle". Most of them gave correct answers in the first part of questionnaire but they could not recognize as a tangent a line that splits the curve, as x-axis in B6, or coincides with a part of it, as the ε_1 in B4. For them, ε was the tangent of the curve in B2, because, as they said in the interviews, the curve looks like a circle, locally. A student was asked during the interview about the tangent of a straight line. Although she knew that the formal definition implies that the tangent at a point is the same line, she replied:

"No, it cannot happen. The straight line does not have a tangent, because the tangent intersects our curve at any neighbourhood around the point"

The third class comprised seven (7/19) students. These participants didn't have any problems to identify a tangent. These students gave correct answers to almost all tasks as they had created a *"curve concept image"* which did not depend on the circle's properties. In the interviews, we asked them to give a formal definition. Only two of them were able to define the tangent at a point as the line which passes through this point and has slope equal to the derivative in this point. All of the students knew that the definition "comes from the derivative" and for this reason they did not care about the validity of the circle's properties. The only criterion for them was: "the point is not corner point". For example one student said:

"I have been taught at school what a tangent line is. I don't remember the formal definition…but I am sure there is not a problem when the line crosses the curve or when the line intersects more than once… but don't ask me why. I don't remember, but I have an intuition which leads me to all the answers that I gave."

DISCUSSION

In order to explain the above findings we have to describe what the students had learnt through their experiences about tangent line. The notion of the tangent line appears in three stages during a student's schooldays. At first, in Euclidean Geometry, students learn the tangent of the circle as a line that has exactly one point common with the circle. An intuitively obvious property of this line is that it has a common point with the circle and divides the plane in two parts, one of which contains the whole circle. Later, in Analytic Geometry, the students are introduced to the conic sections. In these cases, the tangent's definition is more sophisticated: "the tangent in a point A is the limiting position of the secant AB as B approaches A". The "exactly one common point" property remains true in conics, but it is not enough to define the tangent; there are lines, which have one point common with parabola or hyperbola and they are not tangent lines. On the other hand, the "one common point and residence on one semi-plane" property is valid for all cases except hyperbola, where the tangent separates the two branches of it. Consequently, we can say that the property remains true, even in the case of hyperbola, for each branch separately. Therefore, there is no necessity for students to change their previous intuitive images about the two properties of the circle's tangent: "exactly one common point" and "one common point and residence on one semi-plane". A small adaptation of their beliefs is enough to *assimilate* the new knowledge about conics' tangent in their existing knowledge about circle's tangent. In this case, it just needs an enrichment of prior knowledge concerning tangent line.

Finally, in Calculus courses, students encounter the concept of tangent at a point on a curve. At this level, a curve's tangent is defined through the concept of derivative. In fact, this definition is the same as in the case of conic sections. The difference is that none of the above properties remains valid, in general. There are functions that have a tangent that has more than one intersection points with the curve or/and splits the curve into two or more pieces (graph B6).

Analyzing the students' answers, we can say that some of them (first class) use the above properties as the only criterion to identify if a line is tangent. The students of the second class have created a *synthetic model* in their attempts to deal with the tasks of the questionnaire. They know that the general definition of the tangent line does not imply the circle's properties. They also know that the tangent line and its existence, depends on what is happening locally in the curve. Although the two circle's properties are not valid generally, they remain active in their new *concept image* of the tangent line. These students have in mind how a tangent line should "look like". They focus on an area of the curve near the point and test the validity of circle's properties in this part, through their adapted definition. On the other hand, the students of the third group had created an "adequately good" *concept image* of curves' tangent, even though they didn't remember the corresponding *concept definition.* That means that their *concept images* concerning tangent line was not closely dependent on the circle's properties. This is a good basis for their transition to the formal meaning of tangent line especially in cases where the graphical representations become poor in information or they are not trustworthy even in a computational environment like the case of function $f(x) = x^2 \sin \frac{1}{x}$.

Consequently, many of these students have a *concept image* of tangent, involving circle–like pictures. These *concept images* contain a particular representation of tangent that could be called a *generic tangent* (Tall 1986; Vinner 1991). The *generic tangent* acts as a *paradigmatic model* (Fischbein, 1987). It is not an example of the notion of tangent in general but it is a particular exemplar and it is accepted as a representative for the whole class of tangent lines.

In terms of the theory of *conceptual change*, we argue that the ideas related to the notion of the circle's tangent are *beliefs*, which act as a barrier to the process of mastering the notion of curve's tangent. Students usually generate *synthetic models* in their attempts to relate the information they receive about tangent with their knowledge on the circle's properties. This *synthetic model* is a "secondary intuition without formal perfections" that is based on the *paradigmatic model* of circle.

While the number of participants in this study is limited, we believe that it could offer some evidence to support our assumptions. This study suggests that the acquisition of knowledge of tangent line requires a *conceptual change*, which is a complex and discontinuous process. We tend to believe that the main beliefs related to the circle model are "exactly one common point" and "one common point and residence on one semi-plane". These properties are inherited from the circle and they are generic. The circle in this case is prototype and forms their *paradigmatic model* concerning tangent line. These are *secondary intuitions* of students (Fischbein, 1987) but they are not typically correct. They are influenced by their school experience related to circle's properties and they are obstacles to the process of transition to a generalized notion of tangent.

Furthermore, the historical trace of tangent line could give us an interesting point of view of students' difficulties concerning this notion. As Artigue (1991) described, although the first definitions of tangent of circle, firstly, and conics, later on, came too early in the history of mathematics, it was only at sixteenth century when a more general definition of tangent line appeared.

It looks that this transition from tangent of conics to tangent of curve needed a revised thinking about this concept. This innovation in mathematics was not just an addition of new ideas to the previous ones. Actually, it needed the introduction of infinity processes and this was a *revolution in mathematics,* in the sense that Dauben (1984/1992) gave to this notion.

Therefore, a teaching proposal of tangent line could use a revised representation. This approach would take into account the above results concerning *conceptual change* in the case of tangent line and could prepare students from the first stages of the study of this notion for its general features. This could be the local straightness, which is the *cognitive root* for the notion of derivative (Tall, 1989, 2003). The property of *local straightness* refers to the fact that, if we focus close enough to a certain point of a curve, this curve looks like a straight line. Actually, this "straight line" is the tangent line at this point. This property satisfies all cases of tangent lines and it could be facilitated, wherever it is possible, by the use of new technology with appropriately designed software (Tall, 1989, 2003; Giraldo, Calvalho & Tall, 2003).

ACKNOWLEDGMENTS

The research presented in this paper was funded by the University of Athens (ΕΛΚΕ).

REFERENCES

Artigue, M. (1990). Epistémologie et Didactique. *Recherches en Didactique des Mathématiques, 10, 2-3,* 241-286.

Cornu, B. (1991). Limits. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 153-166). Dordrecht, The Netherlands: Kluwer.

Dauben, J. (1984). Conceptual revolutions and the history of mathematics: two studies in the growth of knowledge. Originally appeared in Mendelsohn, E. (Ed.), *Transformation and tradition in the sciences, Essays in honor of I. Bernard Cohen,* (pp.81-103). Cambridge

University Press. (Reprinted in Gilies, D. (Ed.), *Revolutions in mathematics*, (pp.15-20). Oxford University Press, 1992.)

Fischbein, E. (1987). *Intuition in Science and Mathematics: An Educational Approach.* Dordrecht, The Netherlands: Reidel.

Giraldo, V., Calvalho, L. M. and Tall, D.O. (2003). In N.A. Pateman, B.J. Dougherty, & J.T. Zilliox (Eds.), *Proceedings of the 27th PME Conference* (Vol. 2, pp. 445-452). Hawaii.

Hannula, Markku, S., Maijala, H., Pehkonen, E., & Soro, R. (2002) Taking a step to infinity: Student's Confidence with infinity Tasks in School Mathematics. In S.Lehti, & K. Merenluoto (Eds.), *Proceedings of third European Symposium on Conceptual Change. A Process Approach to Conceptual Change* (pp. 195-200). Turku, Finland.

Merenluoto, K., and Lehtinen, E. (2002). Conceptual change in mathematics: Understanding the real numbers. In Limon, M. & Mason, L. (Eds.), *Reconsidering conceptual change: Issues in theory and practice*, (pp.233-258). Dodrecht: Kluwer Academic Publishers.

Stafylidou, S., and Vosniadou, S. (2004). The development of Students' Understanding of the Numerical Value of Fractions. *Special Issue on Conceptual Change. Learning and Instruction 14,* 503–518.

Stavy, R., &Tirosh, D. (2000). *How students (mis)understand science and mathematics: Intuitive rules.* New York: Teachers College Press.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with special reference to limits and continuity. *Educational Studies in Mathematics,* 12, 151-169.

Tall, D.O. (1986). Constructing the Concept Image of a Tangent. *Proceedings of the 11th PME Conference*, Montreal, III, 69-75.

Tall, D.O. (1989). Concept Images, Computers, and Curriculum Change. *For the Learning of Mathematics*, **9**,3 37–42.

Tall, D.O. (2003). Using Technology to Support an Embodied Approach to Learning Concepts in Mathematics. In L.M. Carvalho, & L.C. Guimarães (Eds.), *História e Tecnologia no Ensino da Matemática*, *vol. 1* (pp. 1-28), Rio de Janeiro, Brasil.

Tirosh, D. and Tsamir, P. (2004). An Application of the Conceptual Change Theory to the Comparison of Infinite Sets. In S. Vosniadou, C. Stathopoulou, X. Vamvakoussi, & N. Mamaloukos (Eds.), *Proceedings of the 4th European Symposium on Conceptual Change.* (pp. 96-98) Delfi, Greece.

Vamvakoussi, X., and Vosniadou, S. (2002). Conceptual change in Mathematics: From the set of natural to the set of rational numbers. In S.Lehti, & K. Merenluoto (Eds.), *Proceedings of the Third European Symposium on Conceptual Change. A Process Approach to Conceptual Change*. (pp. 201-204). Turku, Finland.

Vamvakoussi, X., and Vosniadou, S. (2004a). Understanding density: presuppositions, synthetic models and the effect of the number line. In S. Vosniadou, C. Stathopoulou, X. Vamvakoussi, & N. Mamaloukos (Eds.), *Proceedings of the 4th European Symposium on Conceptual Change* (pp. 98-101) Delfi, Greece.

Vamvakoussi, X., and Vosniadou, S. (2004b) Understanding the structure of the set of rational numbers: A conceptual change approach. *Special Issue on Conceptual Change. Learning and Instruction 14,* 453–467.

Van Dooren, W., De Bock, D., Hessels, A., Janssens, D., & Verschaffel, L. (2004). The Illusion of Linearity: A Misconception Requiring Conceptual Change? In S. Vosniadou, C. Stathopoulou, X. Vamvakoussi, & N. Mamaloukos (Eds.), *Proceedings of the 4th European Symposium on Conceptual Change* (pp. 106-108). Delfi, Greece.

Vergnaud, G. (1988). Multiplicative Structures. In J. Heiber, & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141-161). Reston, VA: Lawrence Erlbaum Associates & NCTM

Vergnauld, G. (1990). Epistemology and psychology of mathematics education. In P. Nesher, & J. Kilpatrick (Eds.), *Mathematics and Cognition* (pp. 14-30). Cambridge: Cambridge University Press.

Vinner, S. (1991). The role of definitions in the teaching and learning of Mathematics. In D. Tall (Ed.), *Advanced Mathematical Thinking*, (pp. 65-81). Dordrecht, The Netherlands: Kluwer.

Vosniadou, S. (2004). Extending the conceptual change approach to mathematics learning and teaching. *Special Issue on Conceptual Change. Learning and Instruction 14,* 445-451.

Vosniadou, S. (1994). Capturing and modelling the process of conceptual change. In S. Vosniadou (Guest Editor), *Special Issue on Conceptual Change. Learning and Instruction, 4*, 45-69.

Vosniadou, S. and Brewer, W. F. (1992) Mental Models of the Earth: A Study of Conceptual Change in Childhood. *Cognitive Psychology*, 24, 535-585.