The Embodied, Proceptual and Formal Worlds in the Context of Functions

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Abstract

In this study we use Tall's and his colleagues' theory on mathematical concept development which describes three worlds of operations, the embodied, proceptual and formal world (Tall, in press; Tall, 2003; Watson, Spirou, & Tall, 2002). The purpose of the study is threefold: First, to identify mathematical tasks in the context of function that reflect the three worlds of operations, second to investigate whether students' thinking corresponds to the embodied, the proceptual and the formal modes of thinking, and third, to reveal the structure and relationships among the three worlds of operations as these unfold through students' responses. The study was conducted with first year university students. The results suggested that mathematical tasks can be categorized based on Tall's and his colleagues' theory and indicated that students exhibit different kinds of thinking, which reflect to a large extent the three worlds of operations. Three classes of students were identified in terms of the difficulty level of the tasks: (1) the proceptual, (2) the embodied-proceptual, and (3) the formal. According to Tall's theory the embodied, proceptual, and formal thinking develop in sequence in an individual's life. This study indicates that freshmen university students, who have mathematics as a major in higher secondary school, are only able to deal with the embodied tasks once they had been successful with the proceptual ones. The leap to formal thinking could only be achieved when proceptual manipulations were enhanced with competence in embodied tasks.

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Executive Summary

In this study we use Tall's and his colleagues' theory (Tall's theory) on mathematical concept development which describes three worlds of operations, the embodied, proceptual and formal world (Tall, in press; Tall, 2003; Watson, Spirou, & Tall, 2002). The purpose of the study is threefold: First, to identify mathematical tasks in the context of function that reflect the three worlds of operations, second to investigate whether students' thinking corresponds to the embodied, the proceptual and the formal modes of thinking, and third, to reveal the structure and relationships among the three worlds of operations as these unfold through students' responses.

The study was conducted with first year university students, from the University of Cyprus and the University of Athens, who had mathematics as a major subject in higher secondary school. A test, which included tasks that reflected the characteristics of the three worlds of operations (Tall, in press; Tall, 2003), was administered to 236 students.

The confirmatory factor analysis was applied in order to assess the results of the study. It can be deduced from the analysis of the data presented that mathematical tasks may be categorised based on Tall's and his colleagues' theory, to embodied, proceptual and formal. The mixture growth modelling illustrated that three different classes of students can be identified, which, to a large extent, reflect the "three worlds of operations". Class 1 students were able to respond only to the proceptual tasks. Class 2 students were able to respond to both the proceptual and embodied tasks, while Class 3 students were able to respond to all three types of tasks, the embodied, the proceptual, and the formal.

It was also found that success on any problem by more than 50% of the students in one class was associated with such success by more than 50% in all subsequent classes. The presence of a consistent trend in the difficulty level across the embodied, proceptual and formal worlds supports the existence of a specific developmental trend.

According to the "three worlds theory", growing individuals must progress from the embodied, to the proceptual and finally to the formal way of thinking. However, as soon as any one of these worlds is developed, it may constitute the basis of new growth. This study reveals such a differentiation in the sequence embodied - proceptual–formal. More specifically, the analysis of the data indicated that freshmen mathematics students were more successful in first doing the proceptual tasks and later on the embodied ones in the context of functions. Students were only able to deal with the embodied tasks once they had been successful with the proceptual ones. It seems that the mathematical objects and their properties, such as graphs, could only be understood when students master the mathematical processes represented by symbols. Thus, it can be argued that for freshmen university students, the understanding of functions shifted from proceptual, to embodied and then to formal. The results of this study also suggested that proceptual understanding was not sufficient on its own to lead to formal understanding. The leap to formal thinking could only be achieved when proceptual manipulations were enhanced with competence in embodied tasks.

Introduction

This study is based on Tall's "three worlds theory" on mathematical concept development (Tall, in press; Gray & Tall, 1994; Tall, Gray, Bin Ali, Crowley, DeMarois, McGowen, Pitta, Pinto, Thomas & Yusof, 2000; Gray, & Tall, 2001; Tall, 2003; Watson & Tall, 2002; Watson, Spirou & Tall, 2002). This theory describes three worlds of operations, the embodied, the symbolic-proceptual and the formal; the theory constitutes an attempt to encompass the whole spectrum of students' cognitive development in different mathematical topics from elementary to advanced mathematical thinking.

The present study provides an empirical verification of the three worlds of operations (the embodied, proceptual, and formal) and traces the different types of thinking projected by freshmen university students in the context of function. The paper is organized as follows: First, we briefly describe the theory of the "three worlds" and the theoretical perspectives that relate to the concept of function. We, then, present the methodology and the results of a relevant study conducted in Cyprus and Greece. Finally, the conclusions of the study and their implications for educational planning and teaching are discussed.

Theoretical Background

The "Three World Theory"

Tall and his colleagues (2003), based on the assumption that individuals build up conceptual structures by perception, action and reflection, put forward the theory that these three activities produce three fundamentally different worlds of mathematics, the embodied, the proceptual, and the formal. The embodied world is the basis for all of our activities. It begins with our perceptions and actions on the real world and, through the use of language, it builds from perceptual representations to more abstract representations.

The proceptual world is the world of calculations in arithmetic and the symbolic manipulation in algebra and calculus. It focuses on the symbols which ambiguously describe either a process or a concept. Gray and Tall (1994) used the term 'procept' to describe this dual role of the mathematical symbolism. Procepts can function either as a process (such as addition) or as a concept (such as sum). This special type of symbol can be used either to carry out calculations and manipulations or to think about the mathematical concepts. Later maturity in embodied and proceptual mathematics leads to the recognition of the properties of the concepts until we are able to use these properties as fundamental axioms and definitions in the coherent formal structure of theorems and proof.

Tall and his colleagues suggest that each of the three worlds focuses on different qualities; the embodied world focuses on the objects and their properties, the proceptual world on processes represented by symbols, and the formal on properties and relationships among them. The three modes of operations develop in sequence as the individual progresses through his/her school life. First, the student acts in the embodied world, which has its roots in the kinaesthetic/iconic modes of thought. For example, an individual first understands the number 5 based on his/her perception and actions of five concrete items, i.e., fingers, unifix, dots; then the individual reaches the understanding of number five in its abstract form. Later on, as the individual grows, the proceptual world builds on the embodied actions and the individual can view the symbol 5, either as a collection of five objects or the sum of 3+2, 4+1, etc. Once the understanding of whole number is achieved, more sophisticated concepts may follow such as fractions, decimals, algebra.

Later on, formal thinking develops. Students at this level are able to work on axiomatic systems and definitions, and deduce properties from the axioms and definitions in a sequence of theorems. At this level, thinking is not based on familiar objects but on axioms, which are formulated to define specific mathematical structures. Properties can then be deduced by formal proof to build a number of theorems. For example, for the concept of natural numbers $N_0 = \{0,1,2,...\}$ we accept axiomatically the existence of a set N_0 and a function $s : N_0 \to N_0$ with the following properties:

1. The function *s* is one-to-one.
2. There exists an element of N_0, which we represent with the symbol 0, so that $s(n) \neq 0$, for each element n of N_0.
3. If A is a subset of N_0, it satisfies the properties:
    a) $0 \in A$, and
    b) if $n \in A$, then $s(n) \in A$ for every element n of N_0, then A= N_0.

Based on these axioms we define two operations, the sum and multiplication. All other properties of this set and its operations are deduced from these axioms. However, it needs to be stated that formal thinking is also partly based on students' abilities to observe the properties of the objects of the embodied world and to carry out the processes and manipulation of procepts.

The Concept of Function and the Aims of the Study

The function concept is a complex, multifaceted idea whose power and richness permeate almost all areas of mathematics. In the school curriculum, function is an advanced topic that is typically not explored in detail until the secondary level. Due to the unifying role of the function concept in mathematics and its ability to provide meaningful representations of complex real-world situations (Heid, Choate, Sheets, & Zbiek, 1995), current reform recommendations call for an emphasis on functions to be integrated throughout the school curriculum, beginning in the elementary grades (NCTM, 2000). Given the importance of functions in mathematics and curriculum, it is crucial for researchers to explore the nature of mathematics students' knowledge of functions and especially of freshmen mathematics students.

The understanding of the function concept often appears to be difficult. This difficulty is partly due to the different representations associated with this concept (Christou, Zachariades, & Papageorgiou, 2002; Hitt, 1998), since it may be represented in a visual, verbal or symbolic form. Yerushalmy (1997) has concentrated on students' abilities to deal with these different kinds of representations, and the translation amongst them. Sfard (1992) discussed the difficulty students face in translating and relating algebraic and graphical representations of function. Norman (1992) found that even mathematicians studying for a Masters Degree tend to use just one kind of representation, the graphical one. Christou et al., (2002) identified hierarchical levels among the graphical and symbolic representations of mathematical functions and verified an association between students' ability to identify various representations of the mathematical functions.

Generally, the views of different researchers seem to converge to the idea that most teaching approaches do not make any reference to the relationship and translation between different kinds of representations of the function concept (Yerushalmy, 1997). The different representations and the difficulties that students face, the relationship between the different kinds of thinking that students posses and how these relate to different representations of functions, is not yet clear.

A characteristic of recent developments has been a focus of attention not only on the mathematics to be taught, but also on the mental processes by which the concept of function is conceived and learnt (Tall, 2003). To understand students' mental processes, we need first to realize that there is a spectrum of possible approaches to the concept of function, from real-world functions in which intuitions can be built enactively using visuo-spatial representations, through the numeric, symbolic and graphic representations and on to the formal definitions (Tall, 2003). Students' ability to deal with graphs carries characteristics of visual embodiment, whereas the algebraic manipulation of the function symbolism carries proceptual characteristics (Tall, 2003). It is also possible that an individual may approach the function concept through its definition. However, this is more likely to be demonstrated by experts rather than by secondary school students who often tend to disregard definitions.

In this study, we hypothesize that the concept of function has distinct aspects that represent Tall's three worlds, and we propose to investigate the relationship among these three worlds as it unfolds through the responses of freshmen university students to function tasks. Specifically, the aims of the study were as follows:

(a) To investigate whether different tasks in the context of functions can be categorized as embodied, proceptual and formal.

(b) To trace groups of students that reflect these three modes of thinking.

(c) To examine the structure and relationships amongst these three worlds of operations as they are projected through freshmen mathematics students' responses.

Methodology

Participants and Tasks

Data reported in this paper was collected in 2002 by questionnaires administered to 236 first year students who had mathematics as a major subject in secondary school and were studying at the University of Cyprus and the University of Athens. The tests were administered during the students Calculus course towards the beginning of their first semester.

The questionnaire included 10 tasks. Three tasks aimed to investigate students' abilities in the embodied world, three in the proceptual world and four in the formal world. The problems reflecting the proceptual and embodied worlds were taken from a more extended test that was used in an earlier study by Christou, et al. (2002). For the purposes of the current study the most difficult proceptual and embodied tasks were chosen. For example, Christou et al. (2002) found that the most difficult embodied tasks were the ones that cannot be generated by functions of the form y=f(x) (see tasks E1, E2, and E3 in Table 1). The underlying assumption was that students would have achieved the proceptual and embodied worlds once they were in a position to respond correctly to these tasks.

The problems reflecting the embodied operations, investigated students' abilities to identify the graphs that were generated by functions of the form y=f(x), and the graphs that could not be generated by functions of this form (see tasks E1, E2, and E3 in Table 1). The focus of these tasks is on the graph as an object and on the relative change of y with respect to x and not the underlying symbolic function. Students were not expected to translate and give the symbolic form of these graphs but simply to use the vertical line test, and state whether these are functions of the form y=f(x) or not.

The three tasks (P1, P2, and P3; see Table 1) that aimed to investigate students' abilities to operate in the proceptual world, explored their ability to identify the relationships that define functions, to clearly identify the letter indicating the independent variable, and to provide the largest field of values. Thus the emphasis here is on the manipulation of symbols.

Finally, four tasks were presented in order to investigate students' ability to operate in the formal world. Two of these tasks (see F1, and F2) were simply looking into students' abilities to provide and use definitions of functions. More specifically Task F1 was used in order to determine whether there were any students that were able to answer the proceptual and embodied task without knowing the definition of function. Tasks F3 and F4 required students to reflect on the definition of functions in its abstract form and to identify the properties of functions (see Table 1). Students, in order to respond to these tasks, could not rely on symbolic manipulations or extract information from a given graph.

[Insert Table 1 about here]

Scoring and Analysis

Students' fully correct responses were marked with 1 and the incorrect responses with 0. If a student gave a partly correct response, for example if s/he gave a correct answer but wrong justification, this again was marked with 0. The confirmatory factor analysis (CFA), which is part of a more general class of approaches called structural equation modeling, was applied in order to assess the results of the study. CFA is appropriate in situations where the factors of a set of variables for a given population is already known because of previous research. In the case of the present study, CFA was used to test hypotheses corresponding to the theoretical notion of the three worlds. Specifically, our task was not to determine the factors of a set of variables or to find the pattern of the factor loadings. Instead, our purpose of using CFA was to investigate whether the established theory of the three worlds fits our data.

One of the most widely used structural equation modeling computer programs, MPLUS (Muthen & Muthen, 1998), which is appropriate for discrete variables, was used to test for model fitting in this study. In order to evaluate model fit, three fit indices were computed: The chi-square to its degree of freedom ratio (x^2/df), the comparative fit index (CFI), and the root mean-square error of approximation (RMSEA) (Marcoulides & Schumacker, 1996). The observed values of x^2/df should be less than 2, the values for CFI should be higher than .9, and the RMSEA values should be close to zero. Mplus was also used to trace groups of students who could reflect the embodied, proceptual and the formal world of thinking. To this end we used latent class analysis (LCA), which is part of the mixture growth analysis (Muthen & Muthen, 1998). LCA is a statistical method for finding subtypes of related cases (latent classes) from multivariate data. The results of LCA can also be used to classify cases to their most likely latent class. That is, given a sample of subjects measured on several variables, one wishes to know if there is a small number of basic groups into which cases fall. Once the latent class model is estimated, subjects can be classified to their most likely latent class by means of recruitment probabilities. A recruitment probability is the probability that, for a randomly selected member of a given latent class, a given response pattern will be observed.

Results

The results are presented in relation to the aims of the study. First, we examined the hypothesis, implied by the first aim of the study, i.e., whether the embodied, proceptual and formal worlds constitute distinct representational modes of the function concept. Second, we traced multiple groups of students according to the way in which they solved pre-calculus questions. It was hypothesized, that each group of students could reflect the three worlds of operations, and students' thinking can be categorized along the embodied, proceptual and formal worlds. The latter hypothesis leads to the third aim of the study, i.e., to the examination of the structure of the three worlds, which may indicate the developmental nature of the three worlds. These hypotheses are interwoven and aimed at investigating whether the existing theory stated by Tall and his colleagues can embrace a hierarchy of tasks that may contribute to the instruction of complex mathematical ideas such as functions.

The Distinct Nature of the Three Worlds

From a structural point of view, three factors, the embodied, the proceptual and the formal, should be able to model the performance of students on the tasks addressed Each of the embodied and proceptual factors involved three tasks, while the formal factor involved four tasks. Figure 1 represents the model which best describes Tall's theory of the three worlds of operations. Confirmatory factor analysis (CFA) was used to evaluate the construct validity of the model. CFA showed that each of the tasks employed in the present study loaded adequately (i.e., they were statistically significant since z values were greater than 1.96) on each factor, as shown in Figure 1. It also showed that the observed and theoretical factor structures matched for the data set of the present study and determined the “goodness of fit” of the factor model (CFI=0.946, x^2= 85.134, df= 59, x^2/df=1,4, RMSEA=0.04), indicating that embodied, the proceptual and the formal worlds can represent three distinct function of students’ thinking.

[Insert Figure 1 about here]

Classes of Students and the Developmental Trend

The second aim of the study concerns the extent to which students in the sample vary according to the answers they provided in the test. Specifically, we examined whether there are different types of students in our sample who could reflect the embodied, the proceptual and the formal worlds of operations. Mixture growth modeling was used to answer this question (Muthen & Muthen, 1998), because it enables specification of models in which one model applies to one subset of the data, and another model applies to another set. The modeling here used a stepwise method-that is, the model was tested under the assumption that there are two, three, and four classes of subjects. The best fitting model with the smallest AIC (3417.340) and BIC (3569.749) indices (see Muthen & Muthen, 1998) was the one involving three classes. Taking into consideration the average class probabilities as shown in Table 2, we may conclude that classes are quite distinct, indicating that each class has its own characteristics. The means and standard deviations of each of the three worlds of operations (embodied, proceptual and formal) across the three classes are shown in Table 3.

Table 3 shows that students in Class 3 ($\bar{X}$ =0.69) outperformed students in Class 2 and Class 1 ($\bar{X}$ =0.48, $\bar{X}$ =0.21, respectively) in formal tasks, while students in Class 2 outperformed their counterparts in Class 1. However, the percentage of success of students in Class 1 in all tasks was below 50% showing that these students have difficulties in conceptualizing functions. Class 2 students had difficulties in the formal tasks since their success percentage was lower than 50%. These students were successful in most of the embodied tasks (71%) and solved correctly 66% of the proceptual tasks. Finally, Class 3 students seem to understand not only the embodied and proceptual tasks but also have the ability to think formally (69%).

[Insert Table2 about here]

[Insert Table 3 about here]

From Table 4, which shows the problems solved by more than 50% of the students in each class, it can be deduced that there is a developmental trend in students' abilities to complete the assigned tasks because success on any problem by more than 50% of the students in a class was associated with such success by more than 50% of the students in all subsequent classes. Thus, Class 2, which was the largest (N=109), can be considered as the embodied and proceptual class because this class included the students who successfully solved the embodied and proceptual problems. Class 1 students (N=66) solved two thirds of the proceptual problems and one embodied problem, and thus we could consider Class 1 as the proceptual class. It seems that for some students the links between the proceptual and embodied worlds of operations were not very evident.

As it is shown in Table 4, students belonging to Class 3 can solve function problems using formal thinking and this is the top Class of students. Students in Class 2 systematically solve all embodied and proceptual problems and it is difficult to classify them in any of these two worlds. What distinguishes these students from those in Class 3 is their inability to think formally. Students in Class 1 seem to solve more proceptual than embodied tasks, but their ability in operating in these two worlds is much less than the ability of students in higher classes. This result seems to provide evidence that students are more capable in understanding function problems, which are represented in a proceptual form.

[Insert Table 4 about here]

The Structure of the Developmental Trend

The presence of a consistent trend in the difficulty level across the proceptual, embodied, and formal worlds supports the hypothesis for the existence of a specific developmental trend. The data imply that students firstly grasp the calculus concepts by perceiving them in the proceptual world and secondly by applying the ideas prescribed in the embodied world. The formal world is grasped after the conceptualization of the embodied and proceptual representations. It appears that students in the context of functions, no longer follow the embodied, proceptual, formal development, but the proceptual, embodied, formal sequence. To further examine this sequence, we tested two models for specifying the nature of the developmental trend of students in understanding function problems. The first model assumes that students first understand the concept of function through the embodied world, and then they are able to use proceptual and formal ideas (model 1, see Figure 2). The second model, which results from the data of the present study, assumes that students comprehend the concept of function proceptualy and then they are able to conceive the embodied, and formal representations (model 2, see Figure 2). In both models the formal thinking seems to be the result of students' experience with both the embodied and proceptual worlds.

Structural confirmatory analysis was used to examine the model that best fits the empirical data. From Table 5 and Figure 2, we can deduce that the best model is model 2, since it has the best fitting indices (CFI=.951; x^2= 69.85; df=48; x^2/df=1.46; RMSEA=0.06). Model 1 does not fit the data and all fitting indices are not adequate to provide evidence that supports the structure implied in it (CFI=.859; x^2= 126.777; df=59; x^2/df=2.15; RMSEA=0.09). These results reaffirm the developmental trend as described above and indicate that students are more fluent in doing first the proceptual tasks and later on the embodied ones.

[Insert Table 5 and Figure 2 about here]

Conclusions

The first aim of this study was to examine whether it is possible to identify between different types of tasks that reflect the three worlds of operations. It can be deduced from the data presented in this study that mathematical tasks may be categorized based on Tall's and his colleagues' theory, to embodied, proceptual and formal. The second aim concerned the extent to which students in the sample vary according to the tasks provided in the test. The mixture growth modeling illustrated that three different classes of students can be identified, which, to a large extent, reflect the "three worlds of operations". Class 1 students were able to respond only to the proceptual tasks. Class 2 students were able to respond to both the proceptual and embodied tasks, while Class 3 students were able to respond to all three types of tasks, the embodied, the proceptual and the formal. It was also found that success on any problem by more than 50% of the students in one class was associated with such success by more than

50% in all subsequent classes. The presence of a consistent trend in the difficulty level across the embodied, proceptual and formal worlds supports the existence of a specific developmental trend.

The analysis indicated that students are more successful in first doing the proceptual tasks and later on the embodied ones. Students were only able to deal with the embodied tasks once they had been successful with the proceptual ones. It seems that the mathematical objects and their properties, such as graphs, could only be understood when students master the mathematical processes represented by symbols. Thus, it can be argued that for freshmen university students who have mathematics as a major in higher secondary school the understanding of functions progresses from proceptual, to embodied and then to formal. One of the reasons for this difference may be the emphasis in the mathematics classroom or the nature of the specific mathematical topic. The fact that these students could only understand the graphical representations of functions once manipulation of symbols was achieved does not suggest that embodied understanding is of "less importance". On the contrary, the results of this study suggested that proceptual understanding was not sufficient on its own, to lead to formal understanding. The leap to formal thinking could only be achieved when proceptual manipulations were enhanced with competence in embodied tasks.

A number of teaching implications arise from these findings. Within the theoretical framework of this study, it can be argued that learning in mathematics begins with embodied thinking. As students progress to higher secondary education a "disembodiment" starts taking place, which overshadows embodied thinking in the expense of proceptual thinking. This disembodiment might be due to the teaching approaches in the classroom, which emphasize proceptual ways of thinking in the expense of the embodied ones (Vinner, 1992; Norman, 1992; Yerushalmy, 1997). However, in order to achieve formal thinking it is necessary for a re-embodiment, since proceptual thinking is not sufficient on its own to lead to formal thinking. Students who systematically use only procedures and mathematical symbolism distance themselves from the manipulation of embodied representations and this consequently inhibits their transition to the formal way of thinking.

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