

TEACHING ANALYSIS IN DYNAMIC GEOMETRY ENVIRONMENTS

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In this paper we present instructional activities for the concepts of limit, derivative and definite integral which were designed in the course of a European Comenius 2.1 project. The learning environments are designed so that they approach intuitively, through multiple and interconnected representations, the above mathematical concepts in ways that are consistent with the mathematical theory. Additionally, they take into account students' previous knowledge as well as topics that previous research suggests causing difficulty in calculus courses. The material we present here makes use of dynamic geometry software that offers a function editor / sketch environment as well as tools appropriate for Calculus instruction. Some elements of teachers' feedback in this material are presented.

INTRODUCTION

Functions and Calculus have a wider field of applications in other disciplines and constitute a basic part of the mathematical curriculum of secondary education. In many countries good student performance in pre-Calculus is necessary for university entry. At the same time, as several studies show, the majority of students face serious problems in understanding the concept of function as well as the basic Calculus concepts (for example see Harel, Selden & Selden, 2006).

The work presented in this paper originates in a three-year project called *CalGeo* (Teaching Calculus Using Dynamic Geometric Tools). Amongst the objectives of this project is the design of an in-service teachers' education programme based on learning environments suitable for teaching functions and Calculus in upper secondary education using dynamic geometry tools. The programme focuses on the following topics: introduction to infinite processes, limit, continuity, derivative and integral¹. For each topic the training material includes documentation that raises mathematical, historical and didactical / pedagogical issues and a set of proposed activities. According to the project, the produced material has to be tested in a pilot teachers' training programme as well as in real classroom conditions in each of the participating countries. By the time of the writing of this paper the teachers' training programme has been finished in Greece without any classroom application.

In what follows we first articulate the theoretical assumptions of the project. Then we describe the dynamic environment within which the activities were developed

¹ The countries involved in this project are Greece, Cyprus, England and Bulgaria. The produced material refers to grades 11 and 12 and it is adapted to the Calculus curricula of the participating countries.

accompanied by some results of the pilot training programme applied in Greece. Finally we present some issues for discussion that emerged from this work.

THEORETICAL ASSUMPTIONS

The role of representations in mathematics education has emerged in recent literature as fundamental aspect in facilitating students' construction of mathematical concepts as well as their problem solving abilities (Janvier, 1987; Schoenfeld, 1994; Zimmermann & Cunninham, 1991). Additionally, the technology-based multiple linked representations influence the students' mental constructions of Calculus concepts. For a deeper conceptual understanding of Calculus, instructions should be focused not only on the use of algebraic representations but additionally should take into account the geometric and intuitive representations of the corresponding mathematical objects as well as the interaction among these multiple representations (Kaput 1994). This mode of representing mathematical concepts is gaining more strength due to the advances in computer technology and the development of dynamical mathematical software (Habre & Abboud, 2006).

Mere experience of different representations however does not entail deeper conceptual understanding. All the difficulties could not be explained in terms of a lack of interconnection between representations. Sometimes students' intuitive ideas or informal-spontaneous perceptions influenced by their everyday experience and previous mathematical knowledge block their path towards knowledge acquisition (Cornu, 1991, Fischbein, 1987). These ideas act as epistemological obstacles narrowing students' understanding (Sierpiska, 1994).

The design of a multiple representations learning environment appropriate for Calculus concepts needs to take the considerations expressed by the above literature into account. The activities we describe and discuss in the following section aimed to do so. In this paper we discuss the activities revolving around the definitions of limit, derivative and definite integral.

LEARNING ENVIRONMENT / ACTIVITIES

Each of the following activities was designed in order to be used towards the introduction of a new Calculus concept at upper secondary education level (grade 11 and 12). The activities make use of students' previous knowledge in the following two ways:

- They offer problem solving situations in which previous knowledge will turn out inadequate (e.g. the calculation of the instantaneous velocity or the area defined by a function graph and the x 's axis cannot be done by finite processes as in average velocity or in areas of rectilinear geometrical shapes, respectively).
- They offer the opportunity to explore alternative and generalisable aspects of an already known concept (e.g. the tangent line of the circle as the limiting position of secant lines).

These problems are either historical problems (e.g. Archimedes' calculation of an area or the properties of the tangent line in Euclid's Elements), or modelling of real problems (e.g. problems of calculation of surfaces, motion problems).

The learning environments are designed in order to:

- Approach intuitively the corresponding mathematical notion(s) in ways that are consistent with mathematical theory (e.g. visual representation of the ϵ - δ definition of the limit).
- Take into account the students' previous knowledge (e.g. students' knowledge about circle tangent towards their introduction to derivative).
- Take into account the topics which have proved to be a source of learning difficulties in calculus courses (e.g. difficulties in understanding definitions and their geometric interpretations).
- Embody an intuitive approach of the concept to be taught (e.g. local straightness as the embodied approach of local linearity (Tall, 2003)).
- Offer multiple and interconnected representations of the same concept (e.g. the definite integral as the measurement of an area, as the limit of a sum, as new symbolic expression etc.).

The environments we have developed utilise a dynamic geometry software (DGS) called EucliDraw. In addition to DGS facilities, this software offers a function editor / sketch environment as well as some tools appropriate for Calculus instruction. Indicatively, we refer to the magnification tool that can magnify a specific region of any point on the screen in a separate window. This magnification can be repeated as many times as the user specifies through a magnification factor. Other useful, for Calculus, facilities are these that can partition an interval; construct the lower and upper rectangles covering the area defined by a graph and the x ' x axis; control the number of the decimal numbers of calculations etc.

Activity on the concept of Limit

This activity starts with an instantaneous velocity problem:

Problem: A camera has recorded a 100m race. How could the camera's recording assist in calculating a runner's instantaneous velocity at $T=6\text{sec}$?

The students are familiar with the notion of average speed, through their everyday experience and their school experience in Mechanics. But, for the transition to the calculation of instantaneous velocity understanding the limiting process is essential.

The aims of this activity are the intuitive introduction to the ϵ - δ definition of limit of a function and the connection of numerical and graphical representations towards the clarification of the concept of the limit of a function. Research studies have shown that students often understand the notion of a limit as a dynamic process of "getting close to" a fixed point, often with the perception of "never reaching" the limit; or, as

a certain algebraic procedure to be done and not as a static concept (Cornu, 1991). Since the formal definition is in disagreement with students' dynamic, intuitive ideas of limit, this persistence of their informal ideas means that students are often unable to make sense of the formal definition. Disagreement between formal definitions and informal concepts is only one example of a situation in which a student may hold two mutually contradictory ideas and not notice a conflict. The wording of the question plays a role in the selection of the particular mental image that the student brings into play when presented with a problem. John Monaghan (1991) studied the effects of the language used in teaching and learning limits and he noticed that students should realise how everyday meanings of mathematical phrases can direct them into fallacious interpretations.

The technology-based learning environments offer the facility of graphic representations as well as arithmetic computation. Working in these environments a more balanced approach could be achieved to the concept of limit. This approach could offer the appropriate visualisation with the computer carrying out the calculations internally. In this activity the dynamic geometry based environment, already designed in a file of EucliDraw software, acts in two different roles. In the first part of the worksheet it provides numerical results. This enables students to avoid time consuming calculations. In the second part of the worksheet, it represents the numerical data graphically. The students can then visualize the convergence of function and make the transition to the ε - δ definition. In this environment the axis of the time is displayed (x ' x axis) and the student can change the time t and the fixed time T . The values of the corresponding distance of the runner $s(t)$ and $s(T)$ are calculated as well as the average

speed $\frac{s(T)-s(t)}{T-t}$. In the second part

of the activity, the graph of the average speed function $U(t)$ is sketched. Additionally, the ε zone, an area of length 2ε around L , and the δ zone, an area of length 2δ around T , are displayed ($L=U(T)$). A part of these two zones is coloured in red and green. The green and red colours are used not as a visual effect, but as a tool to represent verbally and graphically complex symbolic expressions.

According to these constructions this part of the graph that includes the points $(t,U(t))$ such as $t \in (T-\delta, T+\delta)$ is in the red zone if $U(t) \notin (L-\varepsilon, L+\varepsilon)$ or in the green square if $U(t) \in (L-\varepsilon, L+\varepsilon)$ (figure 1). The colours of these parts have a semiotic role: the red is the restricted and the green is the permissible area according to the definition.

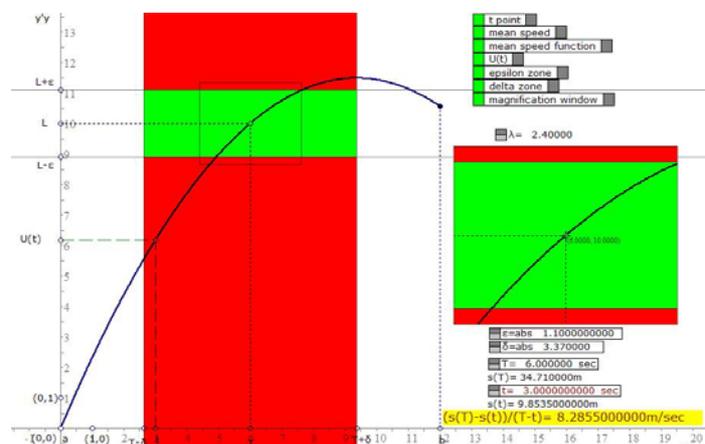


Fig. 1: A EucliDraw environment for the introduction to the notion of limit

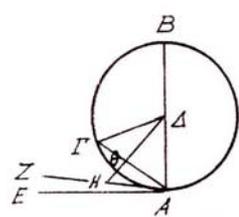
The use of the ε and δ zones in the dynamic geometry environment, gives the student the opportunity to handle in a dynamic way the basic parameters of the problem, in order to understand their relations. The students can change, following the worksheet, the values of ε and look for the appropriate values of δ such as no part of the graph to be in the red region.

The tool of the magnification is used in order to magnify a region of the point (T, L) when the values of ε are too small and the visual perspective is inaccurate.

Activity on the concept of Derivative

This activity starts with a problem inspired by Euclid's "Elements". The students are asked to check the validity of the following proposition (proposition 16 in the third book of 'Elements')²:

"The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilinear angle."



The aims of this activity are: the introduction to the definition of the derivative at a point; the introduction to the definition of the notion of the tangent line of a function curve at a point $(x_0, f(x_0))$ as the linear approximation of the curve at this point; the reconstruction of students' previous knowledge about tangent line grounded to the Euclidean geometry context to general cases of curves; the connection of the symbolic and geometric representations of derivative at a point; the recognition by the students the property of the "smoothness" of a function curve at a point and its relationship to the differentiability of this function at this point.

According to Tall (2003), the cognitive root of the notion of derivative is the local straightness. The property of local straightness refers to the fact that, if we focus close enough to a point of a function curve, in which point the function is differentiable, then this curve looks like a straight line. Actually, this "straight line" is the tangent line of the curve at this point. This property is valid in all cases of tangent lines and it could be facilitated, wherever it is possible, by the use of new technology with appropriately designed software (Tall, 2003; Giraldo & Calvalho, 2006). On the other hand the early experiences of the circle tangent contribute to the creation of a generic tangent as a line that touches the graph at one point only and does not cross it (Vinner 1991). Furthermore, students perceive relevant and irrelevant properties related to the number of common points or the relative position of the tangent line and the graph as defining conditions for a tangent line. Different combinations of these properties create intermediate models of a tangent line. This occurs through the assimilation of

² Translated in English by Heath, (1956, 2, p. 37)

new information about graph tangents in the existing knowledge about circle tangent (Biza, Christou & Zachariades 2006).

This activity starts with the notion of circle tangent. New and more generalisable properties of the tangent line are introduced such as “the best linear approximation of the curve at a point” or “the limiting position of the secant line”. These properties are approached intuitively through the local straightness by the magnification of the curve. These new properties offer an opportunity for a transition to a general definition of a tangent line of a curve and consequently to the introduction of the notion of derivative.

The activity is divided into three steps. The first one is developed in the Euclidean Geometry context. With the help of the magnification tool they magnify the region around A and they observe that the circle looks like its tangent as the magnification factor increases. The students observe that the tangent line is the limiting position of

the secant lines AB as B approaches A . In the second step a transition to the Calculus context is made and students are asked to define a line as a tangent at the case of a function graph. In this step, the EucliDraw file is supposed to be already constructed. In this environment the graph of $y = \sin(x)$ is sketched as well as the point $A(x_0, f(x_0))$. In addition, there are three different buttons that hide the constructions of the following tasks of the exploration. The first one is the “magnification” button that displays the number h , the points $B(x_0+h, f(x_0+h))$ and $C(x_0-h, f(x_0-h))$ and the magnification window of a region of A related to a magnification factor equal to $1/k$. As the h decreases the magnification factor increases and the points B and C are moving closer to A . The second one is the “secant lines” button that displays the secant lines AB and AC . Finally the third one is the “slope” button that displays the slopes of the lines AB and AC (figure 2). In this step students, through different tasks on a worksheet, decrease the number h , observe what has happened with the secant lines and their slope, and they are introduced to the definition of derivative. In the third step students work in the same environment by changing the function to the $y = |\sin(x)|$ that is not differentiable at the points in which the graph meets the x -axis. The students, through the observation on the graph at these points, are introduced to the property of the “smoothness” of a function curve at a point and its relationship to the differentiability.

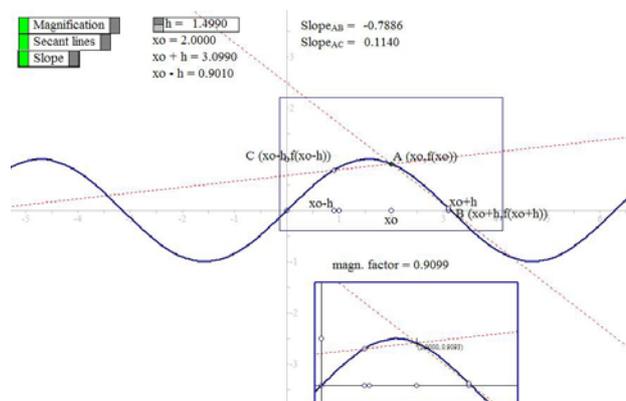
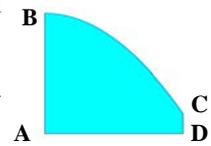


Fig. 2: A EucliDraw environment for the introduction to the notion of derivative

Activity on the concept of Definite Integral

This activity starts with a problem of a parabolic area calculation in order to introduce the students to the notion of Riemann integral:

We are looking for a way to calculate the area of the semi-curved region $ABCD$, which is bounded by three segments AB , AD (x' -axis), DC , and a parabolic segment BC , coming from the graph of a quadratic equation.



Successive calculations of both the upper and the lower Riemann sums are realized in a dynamic geometry environment, for different partitions. Both notions of bound and approximation are essential in the whole process.

The aims of this activity are: the introduction to the notion (and even the definition) of the area of a semi-parabolic region of the plane, through a “natural”, multi-representational way; the intuitive use of Riemann sums in relation to both notions of bound and approximation; reconstruction of the students’ previous beliefs about the notion of measurement, and manipulation of arithmetic, symbolic and geometric representations of the same concept.

Research findings on pre-calculus teaching show that many students, though capable to calculate derivatives or integrals, they do not develop conceptual understanding (Orton, 1983). The activity has the intention to reverse this sequence, and introduce conceptual understanding before or at least in parallel with procedural understanding. The initial problem is the calculation of the area of a semi-parabolic region which can not be achieved by the traditional methods of area calculation. Riemann sums are introduced and used in a “natural” way to extend the traditional notion of area measurement which is useless in this case. Realization and connections with the geometric representation of the whole situation with area counters are supported by the EucliDraw environment. Dynamic tools as the control parameters for the number of covering rectangles and the magnification tool are inevitably used so as to help students understand the meaning of the questions.

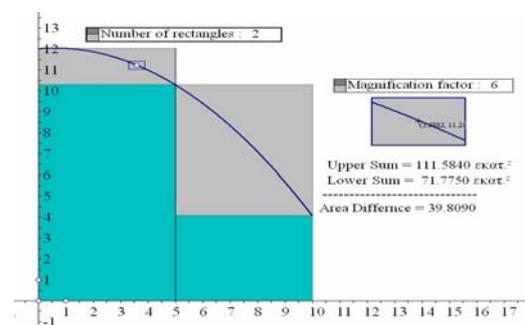


Fig. 3: A EucliDraw environment for the introduction to the notion of definite integral

In this activity the EucliDraw file is supposed to be already constructed. In this environment a parabolic graph is sketched in the domain $[0,10]$ as well as the following constructions: the covering rectangles, both above and under the curve; the parameter n that controls the number of these rectangles; the “upper” and “lower” sums of the rectangles’ areas above and under the curve respectively (are called Riemann sums) and the area difference of these sums. The magnification tool is used to magnify a region of a specific point of the curve as many times as the magnification factor indicates in order to give some sense of the accuracy of covering the parabolic area with rectangles (figure 3).

The design of the activity has taken into account students' previous knowledge about area calculation of rectilinear geometrical shapes. Firstly, the students are asked to think about simpler problems of the same type (e.g. rectilinear geometrical shapes of the plane), having an area which can be easily calculated and to try to apply methods that are already familiar with to the case of parabolic area. Then the students, by using the above described environment and increasing the number of covering rectangles, make, step by step, better approximations for the area in question. If the number of covering rectangles is too big to give a visual perception of the difference between the upper and lower areas, the magnification tool clarifies the visual representation. Students try different values of n , observe the graphic and numeric changes, keep the measurements in a table of their worksheet, and make conjectures based on their observations towards the construction of their knowledge about definite integral.

TRAINING PROGRAMME

The participants of the pilot training programme were 18 Greek mathematics teachers who currently teach in grades 11 and/or 12. Most of the participants had basic computational skills but only a few of them were familiar with mathematics educational software (especial DGS like Geometer's Sketchpad and Cabri) and its application in teaching practices. An introductory course on EuclidDraw software was included in the programme. Although the evaluation of the training programme is still in progress, we can provide some of the teachers' comments and reactions during the programme for each of the activities.

Concerning the limit activity, some of the teachers noticed that the visualization of the effects that ε and δ parameters cause would be helpful, while some others expressed their scepticism on whether the use of this activity will actually assist in overcoming the difficulties they face in their teaching practice (although Greek students are not examined on the ε - δ limit definition, they are presented an informal intuitive introduction to it). In the derivative activity, some teachers commented that the use of Euclid's proposition may contribute to the students' understanding. However, they doubted about the efficiency of students' involvement to *horn-like* angles (defined by the circumference and a straight line). The integral activity was familiar to the teachers as they often use similar problems in order to introduce the notion of integral. Most of the teachers found the dynamic manipulation of the parameters in the environment very useful. Additionally, some of them proposed problems alternative to those of the calculation of the area of a semi-parabolic region.

CLOSING COMMENT

In the wake of recent and rapid technological developments the most challenging question regarding the teaching of Calculus in ways that benefit from these developments is how we devise tools that approach intuitively the mathematical ideas

of Calculus while being consistent with mathematical theory and while taking account the Calculus-related learning issues that mathematics education research has repeatedly highlighted. According to Ferrara, Pratt and Robutti (2006):

... the big revolution in teaching mathematics with technologies was the introduction of dynamicity in software: A dynamic way to control and master the virtual objects on the computer let the student explore many situations and notice what changes and what does not. And the mathematics of change is the first step on the road of calculus. We can intend change at a numerical level, as well as the graphical or symbolic level. (p.257)

In this paper we presented three different activities designed for the introduction to the definitions of limit, derivative and integrals that utilise the above described dynamicity of technology based environments. These activities are a part of a larger piece of work currently in progress. As until now these activities have not been tested in real classroom conditions we wish to confine the aims of this paper to the description of this material and our attempt to integrate the technological progress mentioned above with theoretical underpinnings and concerns made by the relevant mathematics education research. Additionally, we provide some elements of teachers' feedback for the purpose of discussion prior to application.

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